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Existence the Extremal Solution for a Fractional Order Differential Equation

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Abstract: In this paper, we proved the existence the extremal solution for a fractional order differential equation in Banach algebra under lipschitz and Caratheodory conditions via a hybrid fixed point theorem.

Keywords: Banach algebra, fractional order differential equation, existence results.

1. INTRODUCTION

Differential and integral equations are one of the most useful Mathematical tools in both applied and pure mathematics. Moreover the theory of Differential and Integral equations is rapidly developing using the tools of Topology, Functional Analysis and Fixed point theory. This is particularly true of problems in the related fields of Engineering, Mechanical Vibrations and Mathematical Physics. There are numerous applications of differential and integral equations of integer and fractional orders in Electrochemistry, Viscoelasticity, Control theory, Electromagnetism and Porous media etc.

In this paper the work deals with the existence the extremal solution to the following fractional order nonlinear differential equation in Banach Space by Hybrid Fixed Point Theory.

We consider the fractional order differential equations:

$$\left.\begin{aligned} \mathfrak{D}^{\xi}\left[\frac{x(t)}{f\big(t,x(t),x(\gamma(t))\big)}\right] &= g\Big(t,x(t),x\big(\mu(t)\big)\Big),\ t\in\mathbb{R}_+ \\ x(0) &= 0,\ \xi\in(0,1) \end{aligned}\right\} \tag{1.1}$$

Where , $f:\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}-\{0\}$, $g:\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $\gamma,\mu:\mathbb{R}_+\to\mathbb{R}$

By a solution of nonlinear differential equations (1.1) we mean a function $x\in\mathcal{AC}(\mathbb{R}_+,\mathbb{R})$ such that:

(i) The function $t\to\left[\frac{x(t)}{f\big(t,x(t),x(\gamma(t))\big)}\right]$ is absolutely continuous for each $x\in\mathbb{R}$.

(ii) x satisfies (1.1)

2. PRELIMINARIES

In this section we give the definitions, notation, hypothesis and preliminary tools, which will be used in the sequel.

Let $\mathbb{X}=\mathcal{AC}(\mathbb{R}_+,\mathbb{R})$ be the space of absolutely continuous function on $\mathbb{R}_+$ and S be a subset of $\mathbb{X}$. Let a mapping $\mathbb{A}:\mathbb{X}\to\mathbb{X}$ be an operator and consider the following operator equation in $\mathbb{X}$, namely,

$$x(t)=(\mathbb{A}x)(t),\ \text{for all } t\in\mathbb{R}_+ \tag{2.1}$$

Below we give some different characterization of the solutions for operator equation (2.1) on $\mathbb{R}_+$. We need the following definitions.

Definition 2.1[2]: Let $\mathbb{X}$ be a Banach space. A mapping $\mathbb{A}:\mathbb{X}\to\mathbb{X}$ is called Lipschitz if there is a constant $\alpha>0$ such that,

$\|\mathbb{A}x-\mathbb{A}y\|\le\alpha\|x-y\|$ for all $x,y\in\mathbb{X}$. If $\alpha<1$, then $\mathbb{A}$ is called a contraction on $\mathbb{X}$ with the contraction constant α.

Definition 2.2 [2]:.An operator $\mathbb{Q}$ on a Banach space $\mathbb{X}$ into itself is called compact if for any bounded subset $S$of $\mathbb{X}$, $\mathbb{Q}(S)$ is relatively compact subset of $\mathbb{X}$. If $\mathbb{Q}$ is continuous and compact, then it is called completely continuous on $\mathbb{X}$.

Definition 2.3[2]: Let $\mathbb{X}$ be a Banach space with the norm $\|\cdot\|$ and let $\mathbb{Q}: \mathbb{X} \to \mathbb{X}$ be an operator (in general nonlinear). Then $\mathbb{Q}$ is called

i. Compact if $\mathbb{Q}(\mathbb{X})$ is relatively compact subset of $\mathbb{X}$.
ii. Totally compact if $\mathbb{Q}(S)$ is totally bounded subset of $\mathbb{X}$ for any bounded subset $S$of $\mathbb{X}$.
iii. Completely continuous if it is continuous and totally bounded operator on $\mathbb{X}$

It is clear that every compact operator is totally bounded but the converse need not be true.

Definition 2.4 [3]: Let $f \in \mathcal{L}^1[0, \mathbb{T}]$and $\alpha > 0$. The Riemann – Liouville fractional derivative of order ξ of real function f is defined as

$$\mathfrak{D}^{\xi} f(t) = \frac{1}{\Gamma(1-\xi)} \frac{d}{dt} \int_0^t \frac{f(s)}{(t-s)^{\xi}} ds \quad , \quad 0 < \xi < 1$$

Such that $\mathfrak{D}^{-\xi} f(t) = I^{\xi} f(t) = \frac{1}{\Gamma(\xi)} \int_0^t \frac{f(s)}{(t-s)^{1-\xi}} ds$ respectively.

Definition 2.5 [3]: The Riemann-Liouville fractional integral of order $\xi \in (0,1)$ of the function $f \in \mathcal{L}^1[0, \mathbb{T}]$ is defined by the formula:

$$I^{\xi} f(t) = \frac{1}{\Gamma(\xi)} \int_0^t \frac{f(s)}{(t-s)^{1-\xi}} ds \,, \quad t \in [0, \mathbb{T}]$$

Where $\Gamma(\xi)$ denote the Euler gamma function. The Riemann-Liouville fractional derivative operator of order ξ defined by

$$\mathfrak{D}^{\xi} = \frac{d^{\xi}}{dt^{\xi}} = \frac{d}{dt} \circ I^{1-\xi}$$

It may be shown that the fractional integral operator I^{ξ} transforms the space $\mathcal{L}^1(\mathbb{R}_+, \mathbb{R})$ into itself.

Theorem 2.1 [2]: (Arzela-Ascoli Theorem) If every uniformly bounded and equicontinuous sequence $\{f_n\}$ of functions in $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$, then it has a convergent subsequence.

Theorem 2.2 [2]: A metric space X is compact iff every sequence in X has a convergent subsequence.

3. EXISTENCE THEORY

We seek the solution of (2.1) in the space $\mathcal{AC}(\mathbb{R}_+, \mathbb{R})$ of continuous and real – valued function defined on $\mathbb{R}_+$. Define a standard norm $\|\cdot\|$ and a multiplication " $\cdot$ " in $\mathcal{AC}(\mathbb{R}_+, \mathbb{R})$ by,

$$\|x\| = sup\{|x(t)| : t \in \mathbb{R}_+\}, \ \ (xy)(t) = x(t)y(t), \ \ t \in \mathbb{R}_+ \tag{3.1}$$

Clearly, $\mathcal{AC}(\mathbb{R}_+, \mathbb{R})$ becomes a Banach space with respect to the above norm and the multiplication in it. By $\mathcal{L}^1(\mathbb{R}_+, \mathbb{R})$ we denote the space of Lebesgue-integrable function $\mathbb{R}_+$with the norm $\|\cdot\|_{\mathcal{L}^1}$ defined by

$$\|x\|_{\mathcal{L}} = \int_0^{\infty} |x(t)| dt \tag{3.2}$$

Definition 3.1[2]: A mapping $g: \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is Caratheodory if:

i) $t \to g(t, x, y)$ is measurable for each $x, y \in \mathbb{R}$ and
ii) $(x, y) \to g(t, x, y)$ is continuous almost everywhere for $t \in \mathbb{R}_+$.

Furthermore a Caratheodary function g is $\mathcal{L}^1$ −Caratheodary if:

iii) For each real number $r > 0$ there exists a function $h_r \in \mathcal{L}^1(\mathbb{R}_+, \mathbb{R})$ such that $|g(t, x, y)| \leq h_r(t) a.e. \ t \in \mathbb{R}_+$ for all $x, y \in \mathbb{R}$ with $|x|_r \leq r$ and $|y|_r \leq$ r.

Finally a caratheodary function g is $\mathcal{L}^1_{\mathbb{X}}$ −caratheodary if:

iv) There exists a function $h \in \mathcal{L}^1(\mathbb{R}_+, \mathbb{R})$ such that $|g(t, x, y)| \leq h(t), \ \ a.e. \ \ t \in \mathbb{R}_+$ for all $x, y \in \mathbb{R}$

For convenience, the function h is referred to as a bound function for g.

Lemma 3.1: Suppose that $\xi \in (0,1)$ and the function f, g satisfying FODE (1.1) then x is the solution of the FODE (1.1) if and only if it is the solution of integral equation

$$\left.\begin{array}{c} x(t) = \left[f\left(t, x(t), x\big(\gamma(t)\big)\right)\right] \\ \times \left[\dfrac{1}{\Gamma(\xi)} \displaystyle\int_0^t (t-s)^{\xi-1} g\left(s, x(s), x\big(\mu(s)\big)\right) ds\right], t \in R_+ \\ x(0) = 0, \xi \in (0,1) \end{array}\right\} \tag{3.1}$$

Note: The proof of the lemma given in paper [1].

4. EXISTENCE OF EXTREMAL SOLUTION:

Definition 4.1[4] :A closed and non-empty set $\mathbb{K}$ in a Banach Algebra $\mathbb{X}$ is called a cone if

i. $\mathbb{K} + \mathbb{K} \subseteq \mathbb{K}$
ii. $\lambda\mathbb{K} \subseteq \mathbb{K}$ for $\lambda \in \mathbb{R}, \lambda \geq 0$
iii. $\{-\mathbb{K}\} \cap \mathbb{K} = 0$ where 0 is the zero element of $\mathbb{X}$.

and is called positive cone if

iv. $\mathbb{K} \circ \mathbb{K} \subseteq \mathbb{K}$

And the notation $\circ$ is a multiplication composition in $\mathbb{X}$

We introduce an order relation $\leq$ in $\mathbb{X}$ as follows.

Let $x, y \in \mathbb{X}$ then $x \leq y$ if and only if $y - x \in \mathbb{K}$. A cone $\mathbb{K}$ is called normal if the norm $\|\cdot\|$ is monotone increasing on $\mathbb{K}$. It is known that if the cone $\mathbb{K}$ is normal in $\mathbb{X}$ then every order-bounded set in $\mathbb{X}$ is norm-bounded set in $\mathbb{X}$.The details of cone and their properties appear in Guo and Lakshikantham [6].

We equip the space $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$ of continuous real valued function on $\mathbb{R}_+$ with the order relation $\leq$ with the help of cone defined by,

$$\mathbb{K} = \{x \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}): x(t) \geq 0 \ \forall t \in \mathbb{R}_+\} \qquad (4.1)$$

We well known that the cone $\mathbb{K}$ is normal and positive in $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$. As a result of positivity of the cone $\mathbb{K}$ we have:

Lemma 4.1[5]: Let $p_1, p_2, q_1, q_2 \in \mathbb{K}$ be such that $p_1 \leq q_1$ and $p_2 \leq q_2$ then $p_1 p_2 \leq q_1 q_2$.

For any $p, q \in \mathbb{X} = \mathcal{C}(\mathbb{R}_+, \mathbb{R}), p \leq q$ the order interval $[p, q]$ is a set in $\mathbb{X}$ given by,

$$[p, q] = \{x \in \mathbb{X}: p \leq x \leq q\} \qquad (4.2)$$

Definition 4.2 [5]**:** A mapping $G: [p, q] \to \mathbb{X}$ is said to be nondecreasing or monotone increasing if $x \leq y$ implies $Gx \leq Gy$ for all $x, y \in [p, q]$.

For proving the existence of extremal solutions of the equations (1.1) under certain monotonicity conditions by using following fixed pint theorem of Dhage [5]

Theorem 4.1 [5] : Let $\mathbb{K}$ be a cone in Banach Algebra $\mathbb{X}$ and let $[p, q] \in \mathbb{X}$. Suppose that $\mathbb{A}, \mathbb{B}: [p, q] \to \mathbb{K}$ are two operators such that

a. $\mathbb{A}$ is a Lipschitz with Lipschitz constant α,
b. $\mathbb{B}$ is completely continuous,
c. $\mathbb{A}x\mathbb{B}x \in [p, q]$ for each $x \in [p, q]$ and
d. $\mathbb{A}$and$\mathbb{B}$ are nondecreasing.

Further if the cone $\mathbb{K}$ is normal and positive then the operator equation $\mathbb{A}x\mathbb{B}x = x$ has the least and greatest positive solution in $[p, q]$ whenever $\alpha M < 1$, where
$M = \|\mathbb{B}([p, q])\| = sup\{\|\mathbb{B}x\|: x \in [p, q]\}$

Definition 4.3: A function $p \in \mathcal{AC}(\mathbb{R}_+, \mathbb{R})$ is called a **lower solution** of the FODE (1.1) on $\mathbb{R}_+$ if the function $t \to \frac{p(t)}{f\left(t, p(t), p(\gamma(t))\right)}$ is continuous and

$$\left.\begin{array}{c} \mathfrak{D}^{\xi}\left[\dfrac{p(t)}{f\left(t, p(t), p\left(\gamma(t)\right)\right)}\right] \leq g\left(t, p(t), p\left(\mu(t)\right)\right), \\ a.e., t \in \mathbb{R}_+ \\ x(0) = 0 \end{array}\right\}$$

Again a function $q \in \mathcal{AC}(\mathbb{R}_+, \mathbb{R})$ is called an **upper solution** of the FODE (1.1) on $\mathbb{R}_+$ if function the $t \to \frac{q(t)}{f\left(t, q(t), q(\gamma(t))\right)}$ is continuous and

$$\left.\begin{array}{c} \mathfrak{D}^{\xi}\left[\dfrac{q(t)}{f\left(t, q(t), q\left(\gamma(t)\right)\right)}\right] \geq g\left(t, q(t), q\left(\mu(t)\right)\right), \\ a.e., t \in \mathbb{R}_+ \\ x(0) = 0 \end{array}\right\}$$

Definition 4.4: A solution x_M of the FODE (1.1) is said to be maximal if for any other solution x to FODE (1.1) one has $x(t) \leq x_M(t)$ for all $t \in \mathbb{R}_+$. Again a solution x_M of the FODE (1.1) is said to be minimal if $x_M(t) \leq x(t)$ for all $t \in \mathbb{R}_+$ where x is any solution of the FODE (1.1) on $\mathbb{R}_+$.

Definition 4.5 (Caratheodory Case):

We consider the following set of assumptions:

$\mathfrak{B}1)$ g is Caratheodory.

$\mathfrak{B}2)$ The functions $f\left(t,x(t),x\big(\gamma(t)\big)\right)$ and $g\left(t,x(t),x\big(\mu(t)\big)\right)$ are non-decreasing in x almost everywhere for $t\in\mathbb{R}_+$.

$\mathfrak{B}3)$ The FODE (1.1) has a lower solution p and an upper solution q on $\mathbb{R}_+$with $p\le q$.

$\mathfrak{B}4)$ The function $l{:}\,\mathbb{R}_+,\mathbb{R}$ defined by,

$$l(t)=\left|g\left(t,p(t),p\big(\mu(t)\big)\right)\right| + \left|g\left(t,q(t),q\big(\mu(t)\big)\right)\right|$$

is Lebesgue measurable.

Remark 4.1: Assume that $(\mathfrak{B}10-\mathfrak{B}12)$ hold. Then

$$\left|g\left(t,x(t),x\big(\mu(t)\big)\right)\right|\le l(t),a.e.\,t\in\mathbb{R}_+,$$

for all $x\in[p,q]$

and $k\frac{1}{\Gamma(\xi+1)}T^{\xi}\|l\|_{\mathcal{L}^1}<1,\forall t\in\mathbb{R}_+$.

Theorem 4.2 : Suppose that the assumptions $(\mathfrak{B}1)-(\mathfrak{B}4)$ holds and l is given in remark (4.1) then FODE (1.1) has a minimal and maximal positive solution on $\mathbb{R}_+$.

Now FODE (1.1) is equivalent to IE (3.1) on $\mathbb{R}_+$. Let $\mathbb{X}=C(\mathbb{R}_+,\mathbb{R})$ and define an order relation "$\le$" by the cone $\mathbb{K}$ given by (4.1). Clearly $\mathbb{K}$ is a normal cone in $\mathbb{X}$.

Let us define the two mappings $\mathbb{A},\mathbb{B}{:}\,[p,q]\to\mathbb{K}$ by,

$$\mathbb{A}x(t)=\ f\left(t,x(t),x\big(\gamma(t)\big)\right)$$

$$\mathbb{B}x(t)=\frac{1}{\Gamma(\xi)}\int_0^t(t-s)^{\xi-1}g\left(s,x(s),x\big(\mu(s)\big)\right)ds,$$

Thus from the FODE (1.1) we obtain the operator equation as follows:

$$x(t)=\mathbb{A}x(t)\mathbb{B}x(t),\ t\in\mathbb{R}_+ \tag{4.3}$$

We shall show that, the operators $\mathbb{A}$, $\mathbb{B}$ satisfy all the conditions of theorem (4.1)

This will be achieved in the following series of steps.

Notice that $(\mathfrak{B}1)$ implies $\mathbb{A},\mathbb{B}{:}\,[p,q]\to\mathbb{K}$ Since the cone $\mathbb{K}$ in $\mathbb{X}$ is normal, $[p,q]$is a norm bounded set in $\mathbb{X}$. Now it is shown, as in the proof of Theorem (3.1) in the paper [1], that $\mathbb{A}$ is a Lipschitz with a Lipschitz constant $\|\alpha\|$ and $\mathbb{B}$ is completely continuous operator on $[p,q]$. Again the hypothesis $(\mathfrak{B}2)$ implies that $\mathbb{A}$ and $\mathbb{B}$ are non-decreasing on $[p,q]$.

Step I: To show that the operators $\mathbb{A},\mathbb{B}$ are non-decreasing on $[p,q]$. let $x,y\in[p,q]$ be such that $x\le y$.

$$\begin{aligned}\therefore\ \mathbb{A}x(t)&=f\left(t,x(t),x\big(\gamma(t)\big)\right)\\ &\le f\left(t,y(t),y\big(\gamma(t)\big)\right)\\ &\le \mathbb{A}y(t),\forall t\in\mathbb{R}_+\end{aligned}$$

Similarly,

$$\begin{aligned}\mathbb{B}x(t)&=\frac{1}{\Gamma(\xi)}\int_0^t(t-s)^{\xi-1}g\left(s,x(s),x\big(\mu(s)\big)\right)ds\\ &\le\frac{1}{\Gamma(\xi)}\int_0^t(t-s)^{\xi-1}g\left(s,y(s),y\big(\mu(s)\big)\right)ds\\ &\le\mathbb{B}y(t),\forall t\in\mathbb{R}_+\end{aligned}$$

Implies that $\mathbb{A}$ and $\mathbb{B}$ are non-decreasing operators on $[p,q]$.

Step II: Again definition (4.3) and hypothesis $(\mathfrak{B}3)$ implies that ,

$$p(t)\le f\left(t,p(t),p\big(\gamma(t)\big)\right)\times$$

$$\frac{1}{\Gamma(\xi)}\int_0^t(t-s)^{\xi-1}g\left(t,p(t),p\big(\mu(t)\big)\right)ds$$

$$\le f\left(t,x(t),x\big(\gamma(t)\big)\right)\times$$

$$\frac{1}{\Gamma(\xi)}\int_0^t (t-s)^{\xi-1} g\left(s, x(s), x\big(\mu(s)\big)\right) ds$$

$$\leq f\left(t, q(t), q\big(\gamma(t)\big)\right) \times$$

$$\frac{1}{\Gamma(\xi)}\int_0^t (t-s)^{\xi-1} g\left(s, q(s), q\big(\mu(s)\big)\right) ds$$

$\leq q(t), \forall t \in \mathbb{R}_+$ and $x \in [p, q]$

As a result $p(t) \leq \mathbb{A}x(t)\mathbb{B}x(t) \leq q(t), \forall t \in \mathbb{R}_+$ and $x \in [p, q]$

Hence $\mathbb{A}x\mathbb{B}x \in [p, q], \forall x \in [p, q]$

Step III: Again
$M = \|\mathbb{B}([p, q])\| = sup\{\|\mathbb{B}x\| : x \in [p, q]\}$

$$\leq sup\left\{ sup_{t\in\mathbb{R}_+} \int_0^t \frac{1}{\Gamma(\xi)}(t-s)^{\xi-1} \times \left| g\left(s, x(s), x\big(\mu(s)\big)\right) ds \right| : x \in [p, q] \right\}$$

$$\leq sup \frac{1}{\Gamma(\xi)}\left[\frac{(t-s)^{\xi}}{\xi}\right]_0^t \|l\|_{\mathcal{L}^1} \leq \frac{1}{\Gamma(\xi+1)} T^{\xi} \|l\|_{\mathcal{L}^1}$$

Since $\alpha M < K \frac{1}{\Gamma(\xi+1)} T^{\xi} \|l\|_{\mathcal{L}^1} < 1$

We apply theorem (4.1) to the operator equation $\mathbb{A}x\mathbb{B}x = x$ to yield that the FODE (1.1) has minimum and maximum positive solution on $\mathbb{R}_+$.

This completes the proof.

5. CONCLUSION

In this paper we have studied the existence the extremal solution of fractional order nonlinear functional differential equation. The result has been obtained by using hybrid fixed point theorem for two operators in Banach space due to Dhage.

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